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Modular invariance of string theory on AdS_3

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Abstract

We discuss the modular invariance of the $SL(2, R)$ WZW model. In particular, we discuss in detail the modular invariants using the $\hat{sl}(2, R)$ characters based on the discrete unitary series of the $SL(2, R)$ representations. The explicit forms of the corresponding characters are known when no singular vectors appear. We show, for example, that from such characters modular invariants can be obtained only when the level $k < 2$ and infinitely large spins are included. In fact, we give a modular invariant with three variables $Z(z, \tau, u)$ in this case. We also argue that the discrete series characters are not sufficient to construct a modular invariant compatible with the unitarity bound, which was proposed to resolve the ghost problem of the $SL(2, R)$ strings.

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1 Introduction

The string theory on AdS_3 , namely, $SL(2, R)$ is important in various respects and it has been investigated for more than a decade [1]-[8]. It gives us the simplest string model in backgrounds with curved time. Without R-R charges, the system is described by the $SL(2, R)$ WZW model. This WZW model is one of the simplest models of the non-compact CFT. In addition, it is closely related to the string theory in some two ($SL(2, R)/U(1)$) and three (BTZ) dimensional black hole backgrounds. The appearance of the AdS/CFT correspondence [9] aroused the renewed interest in the $SL(2, R)$ WZW model and its Euclidean version, the $SL(2, C)/SU(2)$ WZW model, e.g., [10]-[13].

However, in spite of recent intensive studies, there still remain open questions for these string models at the fundamental level. Such a state of the problem was discussed in [14]. In fact, soon after the study of the $SL(2, R)$ WZW model was initiated, it was found that the model contains negative-norm physical states (ghosts) [1]. So far, there are two types of the proposals for the resolution. In one proposal [2], the discrete unitary series of the $SL(2, R)$ representations is used and it is claimed that ghosts can be removed if the spectrum is truncated so that the spin j and the level k of the current algebra $\hat{sl}(2, R)$ satisfy

$$\frac{1}{2} \leq -j < \frac{k}{2}, \quad k > 2. \quad (1.1)$$

(For details of $\hat{sl}(2, R)$ and our conventions, see the next section.) The condition (1.1) is called the unitarity bound. In the other proposal [5, 6], the principal continuous series and the free field representations of the current algebra are used. However, in both cases, it is still an open question if such proposals are compatible with other consistency conditions of string theory such as the modular invariance and the closure of the operator product expansion. This is because such consistency conditions are not well understood either. Thus it seems important to accumulate precise knowledge about them.

In the literature, there were several arguments about this issue. Regarding the OPE, see [8][11]-[13]. On the modular invariance, for instance, the modular invariants were constructed for the so-called admissible representations in [15, 16]. In [3], it was argued that modular invariants can be obtained for the discrete unitary series by adding the sectors corresponding to some winding modes. In [17], the modular invariants of the $N = 2$ SCFT were expressed in terms of the $\hat{sl}(2, R)$ characters based on the relationship between the $N = 2$ superconformal algebra and $\hat{sl}(2, R)$ [18]. In [6], modular invariants for the principal continuous series were discussed along the line of [5].

In this paper, we discuss in detail the possibility of constructing modular invariants using the discrete series characters, which was also discussed in [14]. We will not include additional sectors as in [3]. In the next section, we review the basics of $SL(2, R)$ and $\hat{sl}(2, R)$ and give the modular transformations of the discrete series characters when no singular vectors

appear. In section 3, we discuss the case in which finite number of the characters are used. In section 4, we discuss the case in which infinite number of the characters without singular vectors are used. In section 5, we give a summary of our results and discuss the implication to the unitarity bound (1.1).

2 Setup

The $SL(2, R)$ current algebra is defined by the commutation relations

$$\begin{aligned} [J_n^0, J_m^0] &= -\frac{1}{2}kn\delta_{n+m}, & [J_n^0, J_m^\pm] &= \pm J_{n+m}^\pm, \\ [J_n^+, J_m^-] &= -2J_{n+m}^0 + kn\delta_{n+m}. \end{aligned} \quad (2.1)$$

The generators of the associated Virasoro algebra are given by

$$L_n = \frac{1}{k-2} \sum_{m \in \mathbf{Z}} : \frac{1}{2} J_{n-m}^+ J_m^- + \frac{1}{2} J_{n-m}^- J_m^+ - J_{n-m}^0 J_m^0 :, \quad (2.2)$$

and its central charge is

$$c = \frac{3k}{k-2}. \quad (2.3)$$

The current algebra $\hat{sl}(2, R)$ contains zero mode subalgebra $SL(2, R)$ generated by J_0^0 and J_0^\pm . In particular, for a given $\hat{sl}(2, R)$ representation V , its ground state subspace $V^0 \equiv \{v \in V | L_0 v = \Delta_j v\}$ with $\Delta_j = -\frac{j(j+1)}{k-2}$ naturally provides a representation of $SL(2, R)$. The operators \vec{J}^2 and J_0^0 act on the states in V^0 as

$$\vec{J}^2 |j, m\rangle = -j(j+1) |j, m\rangle, \quad J_0^0 |j, m\rangle = m |j, m\rangle. \quad (2.4)$$

The other states in V are obtained by acting on $|j, m\rangle \in V^0$ with J_{-n}^a ($n \geq 0$).

On physical grounds, we are interested in the $\hat{sl}(2, R)$ representations V where V^0 are irreducible and unitary as $SL(2, R)$ modules. In such cases, since $-j(j+1)$ is real and invariant under $j \rightarrow -j-1$, one can assume either $j \leq -1/2$ or $j = -1/2 + i\lambda, \lambda \geq 0$ without loss of generality.

For the universal covering group of $SL(2, R)$, there are five classes of such representations:

- (1) Identity representation: the trivial representation with $\vec{J}^2 = J_0^0 = 0$.
- (2) Principal continuous series: representations with $m = m_0 + n$, $0 \leq m_0 < 1$, $n \in \mathbf{Z}$ and $j = -1/2 + i\lambda$, $\lambda > 0$.
- (3) Supplementary series: representations with $m = m_0 + n$, $0 \leq m_0 < 1$, $n \in \mathbf{Z}$ and $\min\{-m_0, m_0 - 1\} < j \leq -1/2$.

- (4) Highest weight discrete series (\mathcal{D}_{hw}): representations with $m = M_{\text{max}} - n$, $n = 0, 1, 2, \dots$, $j = M_{\text{max}} \leq -1/2$ and the highest weight state satisfying $J_0^+ |j, j\rangle = 0$.
- (5) Lowest weight discrete series (\mathcal{D}_{lw}) representations with $m = M_{\text{min}} + n$, $n = 0, 1, 2, \dots$, $j = -M_{\text{min}} \leq -1/2$ and the lowest weight state satisfying $J_0^- |j, -j\rangle = 0$.

If we do not take the universal covering group, the parameters are restricted to $m_0 = 0, 1/2$ in (2), $m_0 = 0$ in (3) and $j = (\text{half integers})$ in (4) and (5).

In the following we will focus on the $\hat{sl}(2, R)$ representations whose ground state subspace corresponds to the discrete series of the type (\mathcal{D}_{hw}) or (\mathcal{D}_{lw}); they will be denoted by V_j^{hw} or V_j^{lw} , respectively.

For, e.g., the $SU(2)$ current algebra, the characters are naturally defined using three variables (see, for example, [19]). With this in mind, we define the irreducible characters by

$$\text{ch}_j(z, \tau, u) \equiv e^{2\pi i k u} \sum_{V_j} e^{-2\pi i J_0^0 z} e^{2\pi i \tau (L_0 - \frac{c}{24})}. \quad (2.5)$$

where the summation is taken over V_j which is the irreducible representation of the type of either V_j^{hw} or V_j^{lw} . The plus sign in the first factor $e^{+2\pi i k u}$ is due to the change $k \rightarrow -k$ compared with the compact case. Since the $SL(2, R)$ current module has infinite degeneracy with respect to L_0 , it is inevitable to keep $z \neq 0$ for the convergence of the characters. In addition, it turns out that, as functions of the only two variables (τ, z) , one cannot obtain proper modular transformations of the characters. Thus the use of the three variables is essential.

To calculate these characters, one needs to know about singular vectors. It is known that the $\hat{sl}(2, R)$ Verma module at level k with the highest weight $|j, j\rangle$ has singular vectors if and only if at least one of the following conditions is satisfied [20]:

$$\begin{aligned} (1) \quad & 2j + 1 = s + (k - 2)(r - 1), \\ (2) \quad & 2j + 1 = -s - r(k - 2), \\ (3) \quad & k - 2 = 0, \end{aligned} \quad (2.6)$$

where r, s are positive integers. For example, in the case of $k - 2 > 0$ and $j \leq -1/2$, the singular vectors appear when $2j + 1 = -s - r(k - 2)$.

Thus, for generic values of j , there are no singular vectors in V_j^{hw} ; the irreducible character ch_j coincides with that of the Verma module: [21, 3]

$$\chi_\mu^{\text{hw}}(z, \tau, u) \equiv e^{2\pi i k u} e^{-2\pi i \mu z} q^{-\frac{\mu^2}{k-2}} \vartheta_1^{-1}(z|\tau). \quad (2.7)$$

Here $q = e^{2\pi i \tau}$, $\mu = j + 1/2$ and

$$\vartheta_1(z|\tau) = 2q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2\pi i z})(1 - q^n e^{-2\pi i z}). \quad (2.8)$$

Similarly, for V_j^{lw} we have $\chi_\mu^{lw}(z, \tau, u) = \chi_\mu^{hw}(-z, \tau, u)$.

So far we have considered \mathcal{D}_{hw} and \mathcal{D}_{lw} with $\mu \equiv j + 1/2 \leq 0$ separately. However with the help of the symmetry $\chi_\mu^{lw}(z, \tau, u) = \chi_\mu^{hw}(-z, \tau, u) = -\chi_{-\mu}^{hw}(z, \tau, u)$, we extend the range of μ as $-\infty < \mu < +\infty$ and drop the superscript hw with the following convention:

$$\chi_\mu(z, \tau, u) \equiv \text{r.h.s. of (2.7)} = \begin{cases} +\chi_\mu^{hw}(z, \tau, u), & (\mu \leq 0) \\ -\chi_{-\mu}^{lw}(z, \tau, u). & (\mu \geq 0) \end{cases}$$

We remark that one cannot consider the specialized characters $\chi_\mu(0, \tau, 0)$ since they diverge in the limit $z \rightarrow 0$ because of the infinite degeneracy with respect to L_0 .

For a special values of $\mu = j + 1/2$ for which the Verma module has singular vectors, the irreducible character ch_j is different from χ_μ . In such a case, the explicit form of ch_j seems unknown except for several cases.

In our normalization of (z, τ, u) , the modular transformations are generated by

$$\begin{aligned} S: \quad (z, \tau, u) &\rightarrow \left(\frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau} \right), \\ T: \quad (z, \tau, u) &\rightarrow (z, \tau + 1, u). \end{aligned} \quad (2.9)$$

Under T -transformation, the characters χ_μ just get phases,

$$\chi_\mu(z, \tau + 1, u) = e^{-2\pi i \left(\frac{\mu^2}{k-2} + \frac{1}{8} \right)} \chi_\mu(z, \tau, u). \quad (2.10)$$

For $k - 2 < 0$, the S -transformation of $\chi_\mu(z, \tau, 0)$ is given in [14]. In our case with three variables, it reads as

$$\chi_\mu\left(\frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau}\right) = \sqrt{\frac{-2}{2-k}} \int_{-\infty}^{\infty} d\nu \, e^{4\pi i \frac{\mu\nu}{k-2}} \chi_\nu(z, \tau, u). \quad (2.11)$$

For $k - 2 > 0$, the right-hand side of (2.11) does not make sense since $\Delta_j \rightarrow -\infty$ as $\nu \rightarrow \pm\infty$ and the integral diverges. Instead, after some calculation, we get a slightly different formula which does converge:

$$\chi_\mu\left(\frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau}\right) = \sqrt{\frac{-2}{k-2}} \int_{-\infty}^{\infty} d\nu \, e^{-4\pi i \frac{\mu\nu}{k-2}} \chi_{i\nu}(z, \tau, u). \quad (2.12)$$

Note that an imaginary $\mu = j + 1/2$ corresponds to a spin of the principal continuous series but $\chi_{i\mu}$ are not the characters for those representations any more.

3 Modular invariants from finite number of discrete series characters

Now let us start the discussion of the modular invariance. In this section, we will discuss the possibility of constructing the modular invariants using finite number of the discrete series characters.

First, we would like to discuss the modular invariants using the characters in generic cases, χ_μ . We then show that *it is impossible to make modular invariants from finite number of χ_μ ; more precisely, a finite sum of the left and right characters*

$$Z(z, \tau, u) \equiv \sum_{\mu, \mu'} N_{\mu, \mu'} \chi_\mu(z, \tau, u) \chi_{\mu'}^*(z, \tau, u), \quad (3.1)$$

with non-zero coefficients $N_{\mu, \mu'}$ cannot be modular invariant. The argument is a simple application of Cardy's for $c > 1$ CFT [22]. We compute in two different ways the leading behavior of Z in the limit $\tau \rightarrow +i0$ with $|z/\tau|$ fixed. On one hand, using the S -transform of ϑ_1 , one finds

$$\begin{aligned} Z(z, \tau, u) &= e^{-4\pi k \operatorname{Im} u} e^{-2\pi \operatorname{Im} \frac{z^2}{\tau}} \left| \tau \vartheta_1^{-2} \left(\frac{z}{\tau} \middle| -\frac{1}{\tau} \right) \right| \sum N_{\mu, \mu'} e^{-2\pi i(\mu z - \mu' z^*)} q^{-\frac{\mu^2}{k-2}} (q^*)^{-\frac{\mu'^2}{k-2}}, \\ &\sim e^{-4\pi k \operatorname{Im} u} \left(\sum N_{\mu, \mu'} \right) \left| \frac{\tau}{4} \sin^{-2}(\pi z/\tau) \right| \tilde{q}^{-1/4}, \end{aligned} \quad (3.2)$$

where $\tilde{q} = e^{-2\pi i/\tau}$. On the other hand, if the partition function is modular invariant, one should get

$$\begin{aligned} Z(z, \tau, u) &= Z\left(\frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau}\right) \\ &\sim e^{-4\pi k \operatorname{Im} u} e^{-2\pi i(\mu_0 \frac{z}{\tau} - \mu'_0 (\frac{z}{\tau})^*)} N_{\mu_0, \mu'_0} \frac{1}{4} \left| \sin^{-2}(\pi z/\tau) \right| \tilde{q}^{-\frac{\mu_0^2 + \mu'^2_0}{k-2} - \frac{1}{4}}, \end{aligned} \quad (3.3)$$

where the pair (μ_0, μ'_0) is chosen so that $(\mu_0^2 + \mu'^2_0)/(k-2)$ takes the maximum value there. Clearly the two behaviors (3.2) and (3.3) cannot be compatible and hence the statement was shown.

Next, we would like to consider more general cases including the representations with singular vectors. In such cases, the characters are not always given by χ_μ and we need to consider the modular invariant partition functions in terms of the irreducible characters ch_j rather than χ_μ :

$$\hat{Z}(z, \tau, u) \equiv \sum_{\mu, \mu'} N_{\mu, \mu'} \operatorname{ch}_j(z, \tau, u) \operatorname{ch}_{j'}^*(z, \tau, u), \quad (3.4)$$

where $N_{\mu, \mu'} > 0$.

We now show that, for $k > 2$, it is impossible to construct modular invariants from finite number of the characters for V_j^{hw} (or V_j^{lw}) only. We prove only the case of V_j^{hw} since the

argument for V_j^{lw} is similar. To this end, we decompose the character $\text{ch}_j^{\text{hw}}(z, \tau, u)$ in terms of the variable z :

$$\text{ch}_j^{\text{hw}}(z, \tau, u) = e^{2\pi i k u} e^{-2\pi i j z} q^{-\frac{\mu^2}{k-2} - \frac{1}{8}} \sum_{p \in \mathbf{Z}} e^{2\pi i p z} \text{ch}_{j,p}(q). \quad (3.5)$$

Note that $\text{ch}_{j,p}(q)$ is a power series in q with non-negative integer coefficients. In the generic case when there are no singular vectors, i.e., $\text{ch}_j^{\text{hw}} = \chi_\mu$, we denote the above expansion by

$$\chi_\mu(z, \tau, u) = e^{2\pi i k u} e^{-2\pi i j z} q^{-\frac{\mu^2}{k-2} - \frac{1}{8}} \sum_{p \in \mathbf{Z}} e^{2\pi i p z} \chi_p(q). \quad (3.6)$$

Comparing this with (2.7) gives

$$\sum_{p \in \mathbf{Z}} e^{2\pi i p z} \chi_p(q) = e^{-\pi i z} q^{1/8} i \vartheta_1^{-1}(z|\tau). \quad (3.7)$$

From the explicit form of the theta function (2.8), one finds that the expansion (3.6) converges absolutely (at least) for

$$0 < \text{Im } z < \text{Im } \tau, \quad \text{Re } z = \text{Re } \tau = 0. \quad (3.8)$$

Since any irreducible character ch_j is obtained by subtracting from χ_μ the contribution from singular vectors, we obtain the inequality

$$0 < \text{ch}_{j,p}(q) \leq \chi_p(q), \quad (3.9)$$

if $\text{Re } \tau = 0$, $\text{Im } \tau > 0$. Therefore one can also evaluate the expansion (3.5) in the region (3.8) and finds that

$$0 < e^{-2\pi i k u} \text{ch}_j^{\text{hw}}(z, \tau, u) \leq e^{-2\pi i k u} \chi_\mu(z, \tau, u). \quad (3.10)$$

From the inequality (3.10), it follows that in the region (3.8)

$$\hat{Z}(z, \tau, u) \leq Z(z, \tau, u). \quad (3.11)$$

If the partition function is modular invariant, i.e., $\hat{Z}(z, \tau, u) = \hat{Z}(\frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau})$, in the limit $\tau \rightarrow +i0$ with $|z/\tau|$ fixed, the above inequality gives

$$e^{-2\pi i(\mu_0 - \mu'_0)z/\tau} F(\frac{z}{\tau}) N_{\mu_0, \mu'_0} \tilde{q}^{-\frac{\mu_0^2 + \mu'^2_0}{k-2} - \frac{1}{4}} \leq \frac{1}{4} \left(\sum N_{\mu, \mu'} \right) \sin^{-2}(\pi z/\tau) | \tau | \tilde{q}^{-1/4}, \quad (3.12)$$

where $F(z/\tau)$ is a function of $e^{2\pi i z/\tau}$ which remains finite in the limit. The inequality (3.12) cannot be satisfied for $k > 2$ and hence the statement was shown.

The above argument cannot be generalized to the case in which the partition function includes the characters for both V_j^{hw} and V_j^{lw} . This is because the series in (2.5) can be

defined in $\text{Im } z \geq 0$ for V_j^{hw} whereas it can be defined in $\text{Im } z \leq 0$ for V_j^{lw} and hence we cannot find the region where the partition function becomes real such as (3.8). In this case, the partition function is not analytic in z unless the analytic continuation is possible. To further discuss this case, we may need to find the explicit expressions of the irreducible characters $\text{ch}_j^{\text{hw(lw)}}$.

One might wonder also if a similar statement holds for $k < 2$. To check this, let us note that the arguments in this section hold for more generic highest and lowest weight representations besides V_j^{hw} and V_j^{lw} since we did not use any specific properties of the unitary representations \mathcal{D}_{hw} and \mathcal{D}_{lw} . However, for $k < 2$, modular invariants using finite number of the characters are actually known [15, 16] for the admissible representations. Thus a simple inequality (3.9) should not exclude the possibility of modular invariants for $k < 2$.

4 Modular invariants from infinite number of discrete series characters

We saw that the possibility of constructing modular invariants from finite number of the discrete series characters is quite limited. In this section we consider whether the infinite sum or the integral of the characters χ_μ leave the possibility of constructing modular invariants.

As we saw in section 2, modular S -transformation of χ_μ is expressed as a superposition of infinitely many characters. However unlike the momentum eigenstates (plane waves) in flat space, it is not clear in what sense $\hat{sl}(2, R)$ characters $\{\chi_\mu\}_{\mu \in R}$ are orthogonal or linearly independent. In particular, the formula for the modular S -transformation of χ_μ might not be unique. In fact, it may be possible to get different expressions by deforming the integration contours in (2.11) and (2.12). Thus let us discuss the possibility of the modular invariants in a definite manner.

Here we would like to show that *the S -transformation of χ_μ cannot be expressed by using the values of μ belonging only to a finite interval, namely, by using χ_μ with $\mu \in [\mu_1, \mu_2]$* . First, let us suppose that

$$\chi_\mu\left(\frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau}\right) = \int_{\nu_1}^{\nu_2} d\nu f(\nu; \mu, k) \chi_\nu(z, \tau, u), \quad (4.1)$$

where $f(\nu; \mu, k)$ is some function which is continuous in $\nu \in [\nu_1, \nu_2]$ and independent of τ . Using the S -transform of ϑ_1 , one finds that (4.1) is modular invariant only if

$$i e^{\pi i(k-2)z^2/(2\tau)} e^{-2\pi i\mu z/\tau} \tilde{q}^{-\frac{\mu^2}{k-2}} = \sqrt{-i\tau} \int_{\nu_1}^{\nu_2} d\nu f(\nu; \mu, k) e^{-2\pi i\nu z} q^{-\frac{\nu^2}{k-2}}. \quad (4.2)$$

This can never happen. Indeed, put $z = 0$ and take the limit $\text{Im } \tau \rightarrow +\infty$. The left hand side tends to 1 whereas the right hand side diverges if $k > 2$ and tends to 0 if $k < 2$.

Similar arguments hold in the above and the following when the integral $\int_{\nu_1}^{\nu_2} d\nu f(\nu; \mu, k)$ is replaced with an infinite sum $\sum_i f(\nu_i; \mu, k)$ with $\nu_i \in [\nu_1, \nu_2]$ as long as the sum makes sense. We will omit discussions in such cases.

Now let us consider a partition function of the following form:

$$Z(z, \tau, u) \equiv \int_I d\mu d\mu' g(\mu, \mu') \chi_\mu(z, \tau, u) \chi_{\mu'}^*(z, \tau, u), \quad (4.3)$$

where $g(\mu, \mu')$ is a weight function continuous on the domain $I = [\mu_1, \mu_2] \times [\mu'_1, \mu'_2]$. It is understood that the measure is zero for the discrete values of μ corresponding to the representations with singular vectors. If such a partition function is modular invariant, it follows that

$$\begin{aligned} |\tau| \int_I d\mu d\mu' g(\mu, \mu') e^{-2\pi i(\mu z - \mu' z^*)} q^{-\frac{\mu^2}{k-2}} (q^*)^{-\frac{\mu'^2}{k-2}} \\ = e^{-\pi(k-2) \operatorname{Im} \frac{z^2}{\tau}} \int_I d\mu d\mu' g(\mu, \mu') e^{-2\pi i(\mu \frac{z}{\tau} - \mu' (\frac{z}{\tau})^*)} \tilde{q}^{-\frac{\mu^2}{k-2}} (\tilde{q}^*)^{-\frac{\mu'^2}{k-2}}. \end{aligned} \quad (4.4)$$

However, since the interval I is finite, a similar argument to the above shows that the equality (4.4) cannot be true. Moreover, a similar argument holds also in the case where $g(\mu, \mu')$ includes distributions (as long as the expression makes sense). Therefore, we have found that *it is impossible to construct modular invariants only from χ_μ with μ belonging to a finite interval $\mu \in [\mu_1, \mu_2]$ even if infinitely many χ_μ are used.*

Thus the modular invariant partition function as a superposition of $\chi_\mu \chi_{\mu'}^*$ is possible only if the partition function contains arbitrarily high spin $|\mu|$. Nevertheless, for $k > 2$, the partition function becomes ill-defined since L_0 spectrum is not bounded from below: $\Delta_j \rightarrow -\infty$ as $|\mu| \rightarrow \infty$. Hence we have reached a conclusion that *for $k > 2$ it is impossible to construct modular invariants only from χ_μ , namely, from the discrete series characters without singular vectors* (if any procedure such as ‘Wick rotation’ is not consistently implemented).

Finally, we consider the partition function for $k < 2$ with no upper bound on the spin $|\mu|$:

$$Z(z, \tau, u) \equiv \int_{-\infty}^{\infty} d\mu d\mu' g(\mu, \mu') \chi_\mu(z, \tau, u) \chi_{\mu'}^*(z, \tau, u), \quad (4.5)$$

To analyze this, we introduce the Fourier transform

$$g(\mu, \mu') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\xi d\xi' e^{-i(\mu\xi - \mu'\xi')} \hat{g}(\xi, \xi'). \quad (4.6)$$

Then for $k - 2 < 0$ one finds that

$$Z\left(\frac{z}{\tau}, -\frac{1}{\tau}, u + \frac{z^2}{4\tau}\right) = \frac{2}{2-k} \int_{-\infty}^{\infty} d\nu d\nu' \hat{g}\left(\frac{4\pi\nu}{k-2}, \frac{4\pi\nu'}{k-2}\right) \chi_\nu(z, \tau, u) \chi_{\nu'}^*(z, \tau, u). \quad (4.7)$$

Thus a sufficient condition of the modular invariance is

$$g(\mu, \mu') = \frac{2}{2-k} \hat{g}\left(\frac{4\pi\mu}{k-2}, \frac{4\pi\mu'}{k-2}\right). \quad (4.8)$$

It is easy to find a solution to this condition. It is just the delta-function,

$$g(\mu, \mu') = \delta(\mu - \mu'), \quad (4.9)$$

and this gives the diagonal partition function,

$$\begin{aligned} Z_{\text{diag}}(z, \tau, u) &= \int_{-\infty}^{\infty} d\mu |\chi_{\mu}|^2 = \int_{-\infty}^0 d\mu (|\chi_{\mu}^{\text{hw}}|^2 + |\chi_{\mu}^{\text{lw}}|^2) \\ &= \frac{1}{2} e^{-4\pi k \text{Im } u} e^{(2-k)\pi \frac{(\text{Im } z)^2}{\text{Im } \tau}} \sqrt{\frac{2-k}{\text{Im } \tau}} |\vartheta^{-2}(z|\tau)|. \end{aligned} \quad (4.10)$$

Here we have used the fact that χ_{μ} with $\mu > 0$ are regarded as $-\chi_{\mu}^{\text{lw}}$ with $\mu < 0$ and recovered the superscripts hw and lw. The diagonal partition function without the variable u was discussed in [14]. In our case, it is straightforward to check that $Z_{\text{diag}}(z, \tau, u)$ is actually modular invariant because of the presence of u .

As pointed out also in [14], $Z_{\text{diag}}(z, \tau, 0)$ was discussed in [7] in the context of a path-integral approach to the $SL(2, C)/SU(2)$ WZW model corresponding to Euclidean AdS_3 . This model has an $\hat{sl}_2 \times \hat{sl}_2^*$ current algebra symmetry and the diagonal partition function may be understood also as the partition function of this model. However, in [7] different spectrum seems to be summed up. It is interesting to consider the precise relationship between the approach here and the one in [7].

5 Discussion

In this paper, we discussed the modular invariants using the discrete series characters. The arguments hold, except for the last one below Eq.(4.5), even if we set $u = 0$ and consider $\text{ch}_j(z, \tau, 0)$ and $\chi_{\mu}(z, \tau, 0)$. Our arguments indicate that the possibility of constructing modular invariants is very limited.

If we use only the characters without singular vectors, the possibility is only in the case where $k < 2$ and χ_{μ} with $|\mu| \rightarrow \infty$ are included. We gave such an example. The resulting modular invariant $Z_{\text{diag}}(z, \tau, u)$ may be regarded as a kind of twisted partition function. However, its physical interpretation is still unclear, in particular, regarding the factor $e^{2\pi i k u}$.

In the case in which the characters with singular vectors are allowed, we showed that one cannot construct modular invariants from finite number of the characters for V_j^{hw} or V_j^{lw} . To complete the analysis, we may need to obtain the explicit forms of the characters with singular vectors.

Nevertheless, it turns out that the case without singular vectors covers physically interesting cases and gives important implication to the unitarity bound (1.1). This is because the condition of the singular vectors (2.6) implies that there are no singular vectors within (1.1). Furthermore, since the spins j in that bound belong to a finite interval, our results indicate

that *one cannot construct modular invariants only from the discrete series characters for the representations satisfying the unitarity bound (1.1)*. This means that one cannot make a consistent string theory on $SL(2, R) = AdS_3$ only from the spectrum from $V_j^{\text{hw(lw)}}$ within (1.1).

Here some comments on the relation to Ref. [14] may be in order. First, we discussed the cases including the representations with singular vectors and our analysis covered the cases of both $k > 2$ and $k < 2$. In addition, we defined the character using three variables as in (2.5). As discussed below (2.5), this was essential to obtain a modular invariant (4.10). Since we carried out a systematic analysis using the asymptotic behavior of the characters, we could derive definite conclusions without any ambiguities concerning which characters are linearly independent. Our analysis thus pinned down when the modular invariants can be constructed.

Since there exist ghosts for the discrete series outside the unitarity bound, simply adding such spectrum may not give a consistent theory. Therefore, the possibilities for a consistent theory seem (a) to use the discrete series satisfying (1.1) but include some new sectors with different characters from χ_μ as in [3], and/or (b) to use the spectrum of other representations as in [5, 6]. In any case, the string theory on AdS_3 definitely deserves further investigations.

Note added

While we were proofreading the manuscript, a paper [23] appeared which discusses the modular invariance using the discrete series \mathcal{D}_{hw} and \mathcal{D}_{lw} . In [23], the diagonal modular invariant (4.10) is obtained from the spectrum satisfying the (more stringent) unitarity bound by (i) including additional sectors along the line of [3] (i.e., the possibility (a) in section 5 in our paper) and by assuming ‘Wick rotation’ (see the comment in section 4). In addition, the role of $e^{2\pi i k(u-u^*)}$ in $Z_{\text{diag}}(z, \tau, u)$ in our paper is played by the chiral anomaly term $e^{\pi k \frac{(\text{Im } z)^2}{\text{Im } \tau}}$ in [23]. A relationship to the $SL(2, C)/SU(2)$ case [7] is also discussed there.

The authors of [23] argue that the Hilbert space of the consistent $SL(2, R)$ string theory consists of the principal continuous series, \mathcal{D}_{hw} and \mathcal{D}_{lw} including the sectors generated by the Weyl reflections. This is consistent with the discussion in section 5.

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